# Grade 7/8 Math Circles
## Nov 7/8/9/10, 2022
## Induction - Lesson

## Introduction

**Example 1**

In Rainland, if it rains on one day, then it will always rain the next day. Suppose it is raining today.

- Will it rain tomorrow?
- Will it rain in 10 days?
- Will it rain in 1000 days?

**Example 1 Solution**

Since it is raining today, it will rain tomorrow. Since it will rain tomorrow, it will rain in 2 days. And so on... Let's summarize this in a table:

| Day | Will it rain in Rainland? | Justification |
|---|---|---|
| Today | Yes | Given in question |
| Tomorrow | Yes | It is raining today |
| In 2 days | Yes | It will rain tomorrow |
| ... | ... | ... |
| In 9 days | Yes | It will rain in 8 days |
| In 10 days | Yes | It will rain in 9 days |
| ... | ... | ... |
| In 999 days | Yes | It will rain in 998 days |
| In 1000 days | Yes | It will rain in 999 days |
| ... | ... | ... |

Notice that there will never be a day without rain in Rainland since we will always be able to trace back the rainy days to today.

This is an example of what we call **induction**, a method of proving that a mathematical statement is true for all **natural numbers**.

**Background**

In order to formally state the **Principle of Mathematical Induction**, we first need to introduce some background concepts:

- The **natural numbers**, represented using the symbol $\mathbb{N}$, is the set of positive whole numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$$

  We write $n \in \mathbb{N}$ if $n$ is a natural number and $n \notin \mathbb{N}$ if $n$ is not a natural number. For example, $1 \in \mathbb{N}$ and $234 \in \mathbb{N}$, but $0 \notin \mathbb{N}$ and $4.6 \notin \mathbb{N}$.

- A **mathematical statement** is a sentence that is either true or false. 'The sky is beautiful today' is not a mathematical statement because it's an opinion. However, '5 is a whole number' or '$2 = 9$' are both mathematical statements: the former is true and the latter is false. We typically just use the word 'statement' to refer to a mathematical statement.

- Let $P(n)$ be a statement which depends on a natural number $n$. Consider the statement 'For all $n \in \mathbb{N}$, $P(n)$'. For this statement to be true, $P(n)$ must be true for every natural number $n$. If there is a single natural number $n$ for which $P(n)$ is false, then 'For all $n \in \mathbb{N}$, $P(n)$' is false.

- For example, Goldbach's conjecture states that 'every even natural number greater than 2 can be written as the sum of two prime numbers'. This has been verified to be true for all natural numbers less than $4 \times 10^{18}$ (four quintillions) but it is still not considered true because it has yet to be proven for all even natural numbers.

**Exercise 1**

For each of the following statements, determine whether it is True or False and provide justification.

1. For all $n \in \mathbb{N}$, $n - 1 \in \mathbb{N}$.
2. For all $n \in \mathbb{N}$, $n + (n + 1)$ is odd.
3. For all $n \in \mathbb{N}$, $n + 1 \leq 2n$.
4. For all $n \in \mathbb{N}$, $n^2 + 1 \leq 100n$.
5. For all $n \in \mathbb{N}$, $n + 1 \in \mathbb{N}$.

**Exercise 1 Solution**

1. False: if $n = 1$, $1 - 1 = 0$ is not a natural number.
2. True: for any $n \in \mathbb{N}$, $n + (n + 1) = 2n + 1$. Since $2n$ is even, $2n + 1$ is odd.
3. True: $n + 1 \leq 2n$ is equivalent to $1 \leq n$ (subtract $n$ from both sides). This is true for all $n \in \mathbb{N}$ (by definition of the natural numbers).
4. False: when $n = 100$, we get $100^2 + 1 \leq 100^2$ which is false.
5. True: if $n$ is a positive whole number, then so is $n + 1$. Therefore the statement is true by definition of the natural numbers.

## Principle of Mathematical Induction

In mathematics, a **proof** is a method of communicating mathematical thinking. More specifically, it is a logical argument which explains why a statement is true or false.

**Principle of Mathematical Induction**

Let $P(n)$ be a statement which depends on a variable $n \in \mathbb{N}$ (e.g., in Example 1, $P(n)$ was 'it will rain in $n$ days'). If,

1. $P(1)$ is true.
2. For any $k \geq 1$, if $P(k)$ is true, then $P(k + 1)$ is also true.

Then $P(n)$ is true for all natural numbers $n$.

We can use induction to prove that $P(n)$ is true for all natural numbers $n$ by writing a four-part proof:

- **Base Case**: Prove that $P(1)$ is true.
- **Inductive Hypothesis**: Let $k \in \mathbb{N}$ and assume that $P(k)$ is true.
- **Inductive Step**: Using $P(k)$, prove that $P(k + 1)$ is true.
- **Conclusion**: By the Principle of Mathematical Induction, conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

**Example 1 Revisited**

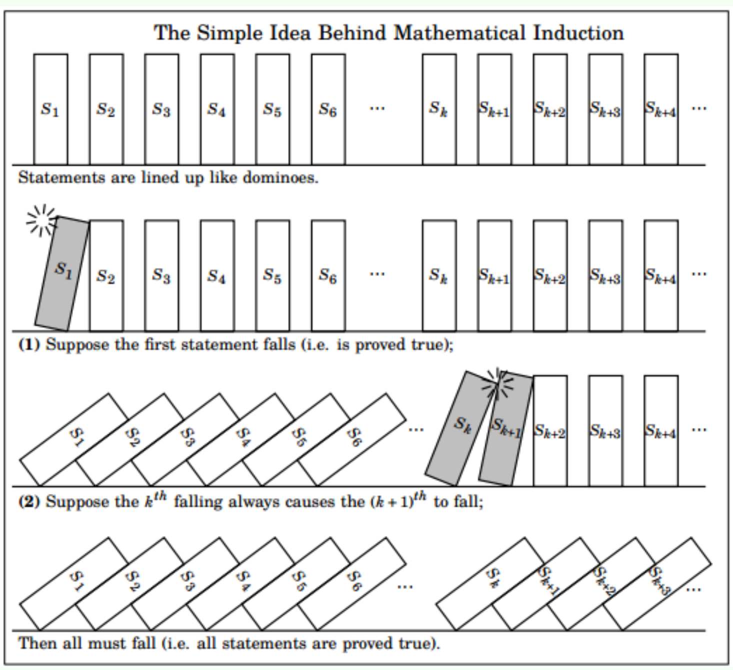
- **Base case**: It will rain today, therefore it will rain tomorrow ($P(1)$ *is true*).
- **Inductive Hypothesis**: Let $k \in \mathbb{N}$ and assume that it will rain in $k$ days (*assume* $P(k)$ *is true*).
- **Inductive Step**: By the rules of Rainland, since it will rain in $k$ days, it will rain in $k+1$ days ($P(k)$ *implies* $P(k+1)$ *is true*).
- **Conclusion**: By the Principle of Mathematical Induction, we can conclude that, for any $n \in \mathbb{N}$, it will rain in $n$ days (*therefore* $P(n)$ *is true for all natural numbers* $n$).

Therefore there will never be a day where it doesn't rain.

Breaking up the proof into the different steps makes it easier for the reader, and also decreases the likelihood of mistakes in the logic, especially for more complicated proofs.

**Example 2: the Domino analogy**

Suppose we have an infinite number of dominoes lined up, starting at a certain point. If we tip the first domino, then it will cause the next one to fall, which will cause the next one to fall, and so on... Thus, every domino will fall:




Source: Taken from Richard Hammack's *Book of Proof*

Induction is considered to be an axiom. That is, the Principle of Mathematical Induction is 'self-evident', and does not require a proof.

**Review of exponents**

If $a$ is a real number (a number on the number line) and $b$ is a natural number, then

$$a^b = \underbrace{a \times a \times \ldots \times a}_{b \text{ times}}$$

For example,

- $2^7 = \underbrace{2 \times 2 \times \ldots \times 2}_{7 \text{ times}}$
- $(4.167)^4 = 4.167 \times 4.167 \times 4.167 \times 4.167$
- $100^1 = 100$

One useful property of exponents is that, for all real numbers $a$ and natural numbers $b$ and $c$,

$$\begin{aligned} a^b \times a^c &= \underbrace{a \times a \times \ldots \times a}_{b \text{ times}} \times \underbrace{a \times a \times \ldots \times a}_{c \text{ times}} \\ &= \underbrace{a \times a \times \ldots \times a}_{b+c \text{ times}} \\ &= a^{b+c} \end{aligned}$$

Notably, $a^b = a^1 \times a^{b-1} = a \times a^{b-1}$.

**Example 3**

Use induction to prove that $n + 1 \leq 2^n$ for all $n \in \mathbb{N}$.

**Example 3 solution**

Let $P(n)$ be $n + 1 \leq 2^n$.

- Base Case: when $n = 1$, $1 + 1 \leq 2^1$ is true so $P(1)$ is true.
- Inductive Hypothesis: Let $k$ be a natural number and assume $P(k)$: $k + 1 \leq 2^k$.
- Inductive Step: We need to prove $P(k + 1)$: $(k + 1) + 1 \leq 2^{k+1}$.

First, we can add 1 to both sides of $P(k)$ to make the left side of the inequality $(k+1)+1$:

$$(k+1)+1 \leq 2^k + 1$$

Since $2^k$ is at least 1, we obtain

$$2^k + 1 \leq 2^k + 2^k = 2 \times 2^k = 2^{k+1}$$

Therefore, combining these two inequalities, we get

$$(k+1)+1 \leq 2^k + 1 \leq 2^{k+1}$$

so $P(k+1)$ is true.

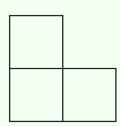
- Conclusion: By the Principle of Mathematical Induction, $n+1 \leq 2^n$ for all $n \in \mathbb{N}$.

Whenever we approach a problem using induction, we should always ask ourselves: how can the statement $P(k)$ be applied to prove $P(k+1)$? The tricky part of proofs is not arriving at an answer (we already know the statement we need to prove), but rather finding the steps in between.

### A Harder Induction Problem

**Example 4**

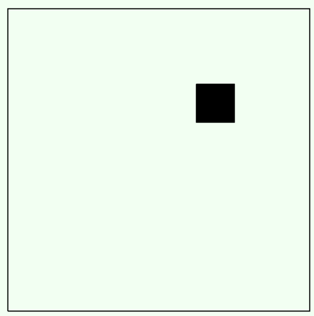
Prove that, for all $n \in \mathbb{N}$, every $2^n$ by $2^n$ grid of squares with exactly one square removed can be covered by triominoes (without overlap). A triomino is an $L$-shaped tile with 3 squares:



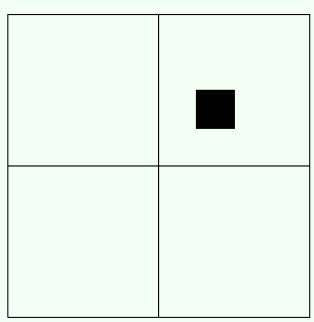
**Example 4 Solution**

We prove this result by induction on $n$, where $P(n)$ is 'all $2^n \times 2^n$ grids of squares with one square removed can be covered by triominoes'.

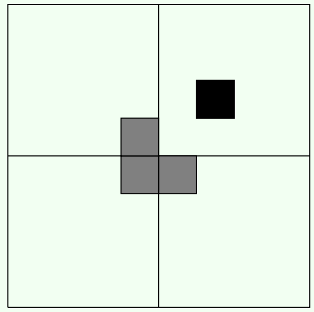
- **Base Case**: The statement $P(1)$ is given by: All $2 \times 2$ grids of squares with one square removed can be covered by triominoes. All $2 \times 2$ grids with one square removed are given by  or  or  or . They can all be covered by a single triomino, rotated by either $0°$, $90°$, $180°$ or $270°$. Therefore, $P(1)$ is true.
- **Inductive Hypothesis**: Let $k \in \mathbb{N}$ and assume that $P(k)$ is true: all $2^k \times 2^k$ grids of squares with one square removed can be covered by triominoes. Note this includes every possible position for the empty square within the grid.
- **Inductive Step**: We wish to prove $P(k+1)$, which stands for: All $2^{k+1} \times 2^{k+1}$ grids of squares with one square removed can be covered by triominoes. Consider a $2^{k+1} \times 2^{k+1}$ grid with one square removed (where the missing square is drawn in black):



Our inductive hypothesis can only be used on $2^k \times 2^k$ grids, so we first need to ask ourselves: how are $2^k \times 2^k$ grids related to $2^{k+1} \times 2^{k+1}$ grids?
We know $2^{k+1} = 2(2^k) = 2^k + 2^k$. Thus, we can split our $2^{k+1} \times 2^{k+1}$ grid in half, horizontally and vertically, to obtain four $2^k \times 2^k$ subgrids:



The missing square occurs in one of these four subgrids. Now, to start covering the grid by triominoes, we'll place one tile around the centre of the grid, covering a corner square in each of the three $2^k \times 2^k$ subgrids that do not contain the missing square:



We can now view the grid as being made up of four $2^k \times 2^k$ subgrids, each with one square missing. The Inductive Hypothesis tells us that each of these four grids can be covered by triominoes. Together with the initial triomino in the centre, this means that the $2^{k+1} \times 2^{k+1}$ grid (with the missing black square) can be covered by triominoes. This proves that $P(k+1)$ is true.

- **Conclusion**: By the Principle of Mathematical Induction, we can conclude that for any $n \in \mathbb{N}$, every $2^n \times 2^n$ grid with one square removed can be covered by triominoes.

### Induction to Prove Summation Formulas

**Example 5**

Using induction, prove that $1 + 2 + ... + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

**Example 5 solution**

Let $P(n)$ be the statement $1 + 2 + ... + 2^n = 2^{n+1} - 1$.

- Base Case: when $n = 1$, $1 + 2 = 2^2 - 1$ is true, therefore $P(1)$ is true.
- Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume $P(k)$ is true: $1 + 2 + ... + 2^k = 2^{k+1} - 1$.
- Inductive Step: using $P(k)$,

$$\begin{aligned} 1 + 2 + ... + 2^k + 2^{k+1} &= (1 + 2 + ... + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

therefore $P(k+1)$ is true.

- Conclusion: $1 + 2 + ... + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Induction provides us a nice way to prove summation formulas because, as seen above, we can use the inductive hypothesis to reduce a summation to a more approachable expression.

**Exercise 2**

Use induction to prove that the sum of the first $n$ odd natural numbers is $n^2$. That is, prove that

$$1 + 3 + 5 + ... + (2n-1) = n^2$$

for all $n \in \mathbb{N}$.

*Hint: For all* $n \in \mathbb{N}$, $(n+1)^2 = n^2 + 2n + 1$.

**Exercise 2 Solution**

Let $P(n)$ be the statement $1 + 3 + 5 + ... + (2n-1) = n^2$.

- Base Case: when $n = 1$, $1 = 1^2$ therefore $P(1)$ is true.
- Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume $P(k)$ is true: $1 + 3 + ... + (2k-1) = k^2$.
- Inductive Step: using $P(k)$ and the hint,

$$\begin{aligned} 1 + 3 + ... + (2k-1) + (2k+1) &= (1 + 3 + ... + (2k-1)) + (2k+1) \\ &= (k^2) + (2k+1) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

  Therefore $P(k+1)$ is also true.

- Conclusion: $1 + 3 + 5 + ... + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

## Strong Induction

Recall the domino analogy: 'if one domino falls, then the next one will also fall'. However, if we start by tipping the first domino, by the time we'll have reached domino $k + 1$ (for some $k \in \mathbb{N}$), then dominoes 1, 2, ..., $k - 1$, and $k$ will all have fallen. Therefore, we can strengthen our induction hypothesis without affecting the underlying logic.

**Strong Induction**

Let $P(n)$ be a statement which depends on a variable $n \in \mathbb{N}$. If,

1. $P(1)$ is true.
2. For all $k \geq 1$, if $P(1), P(2), ..., P(k-1), P(k)$ are all true, then $P(k+1)$ is also true.

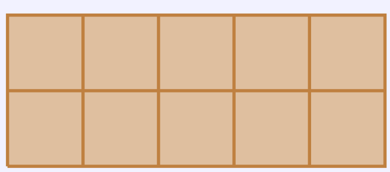
Then $P(n)$ is true for all $n \in \mathbb{N}$.

**Stop and Think**

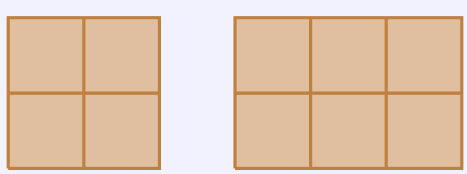
Can we use the Principle of Mathematical Induction to prove Strong Induction?

**Exercise 3**

A chocolate bar is a rectangular grid of chocolate squares. We can break any chocolate bar by making one straight horizontal or vertical break (along the grid lines), separating the bar into two smaller chocolate bars. For Example, if we had a $2 \times 5$ chocolate bar:



we could break it into a $2 \times 2$ bar and $2 \times 3$ bar:



If we start with a chocolate bar with $N$ squares (in the above example $N = 2 \times 5 = 10$), how many breaks will it take to obtain $N$ individual $1 \times 1$ pieces? *Hint: use strong induction on $N$*

**Exercise 3 Solution**

No matter how we break the chocolate bar, it will always take $N-1$ breaks to separate the bar into individual squares. We use strong induction on $N$ to show this. Let $P(N)$ be the statement: every chocolate bar with $N$ pieces take $N-1$ breaks to separate into individual squares.

- **Base Case**: if $N = 1$, the chocolate bar a $1 \times 1$ square and so we will require 0. Therefore, $P(1)$ is true.
- **Inductive Hypothesis**: Let $k \in \mathbb{N}$ and assume $P(k')$ is true for all $k' \leq k$. That is, for all $k' \leq k$, assume that any chocolate bar with $k'$ pieces will take $k'-1$ breaks to separate into individual squares.
- **Inductive Step**: Consider the first time we break a $k+1$ piece chocolate bar. We will obtain two resulting bars, with $A$ and $B$ pieces. Note $A + B = k + 1$ since we do not change the amount of chocolate. By our inductive hypothesis, it will take $A - 1$ breaks to separate the first chocolate bar into individual squares, and $B - 1$ breaks to separate the second chocolate bar into individual squares. Therefore, including the first break we made, there will be a total of

  $$(A-1) + (B-1) + 1 = (A+B) - 1 = (k+1) - 1$$

  breaks. Since our first break was arbitrary, this will be true no matter how we break the chocolate bar.
- **Conclusion**: By Strong Induction, we can conclude that any chocolate bar with $N$ squares must be broken exactly $N-1$ times in order to be separated into individual $1 \times 1$ squares.